Modelling Periodic Behaviour

What do the sounds of your favourite band, the idling of a car engine, the phases of the moon, and your heartbeat all have in common? All are examples of processes that repeat in a regular pattern. Your heart, when you are at rest, follows the same cycle each time it beats. A car engine has moving parts that repeat the same motions over and over. In the lunar cycle, the moon grows from a tiny sliver, to a beautiful full moon, and then wanes until it disappears, only to reappear at the beginning of the next cycle. Your favourite band plays instruments that create repetitive pressure patterns in the air that your ears interpret as music. Periodic patterns can be modelled by trigonometric functions.

Investigate

How can you model periodic behaviour mathematically?

Antique carousels featuring intricately painted horses and other animals are still popular attractions. Several towns in Ontario feature them, as do many amusement parks. Suppose that a carousel has a diameter of 10 m. John is standing at the edge of the carousel, watching his sister Suzanne on a horse on the carousel’s outer edge. How does the distance between John and Suzanne change as the carousel completes a full turn? Predict the shape of a graph that represents distance versus angle for one revolution.

1. Draw a circle to represent the carousel. Mark a point just outside the circle to represent John. Mark a point on the circumference to represent Suzanne at the point where she passes John.

2. Assume that the carousel turns in a counterclockwise direction and that the distance between John and the edge of the carousel can be ignored. Mark points on the circle to represent Suzanne’s position at 30° intervals for one complete revolution. For each position, determine the distance between John and Suzanne using appropriate trigonometric tools. (The diagram illustrates the first three distances to measure.) Record your results in a table with angle in the first column and distance, in metres, in the second column.
3. **Reflect** Predict the values for Suzanne’s position as the carousel continues to rotate through 360° to 720°. Justify your predictions.

4. Use the values in your table to sketch the graph of distance versus angle of revolution. Record distance along the vertical axis and the angle of revolution from 0° to 720° along the horizontal axis.

5. **Reflect** Compare the graph to your predicted graph. How are the graphs similar? How are they different?

6. a) Inspect the graph for two revolutions of the carousel. Predict the total angle of revolution that Suzanne moves through in five revolutions.

   b) If a ride consists of 12 revolutions, what is the total angle of revolution that Suzanne moves through?

7. a) Use the graph to estimate two angles during the first revolution when the distance between John and Suzanne is 8 m. Locate these angles on your diagram of the carousel.

   b) In the third revolution, predict the angles when the distance between John and Suzanne is 8 m.

8. How many **cycles** are shown in the graph?

9. The **period** of a pattern is measured in units appropriate to the problem. What is the period of this pattern?

10. **Reflect** How does a **periodic function** differ from a linear function or a quadratic function?

11. What is the minimum distance between John and Suzanne during the first revolution? What is the maximum distance?

12. What is the **amplitude** of this function?

13. **Reflect** Suppose that Suzanne is on a horse that is 2 m from the centre of the carousel. Predict how the graph will change from the one that you drew in step 4. Sketch your prediction of the graph of distance versus angle of revolution for two revolutions of the carousel.
Example 1

Classify Functions

a) Examine each graph. Determine whether the function is periodic. If it is, determine the period.

i) A periodic function has a pattern of \(y\)-values that repeats at regular intervals. The period is the length of the interval. In this example, the pattern of \(y\)-values in one section of the graph repeats in the next section. Therefore, the function is periodic.

To determine the period, select a convenient starting point and note the \(x\)-coordinate. In this case, choose \((-6, 0)\). Move to the right, and estimate the \(x\)-coordinate where the next cycle begins. This appears to be at the origin. Subtract the two \(x\)-coordinates. The period is \(0 - (-6)\), or 6 units.

ii) In this example, the pattern of \(y\)-values in one section of the graph does not repeat in the next section. Therefore, the function is not periodic.

b) Examine the graph. Determine whether the function is periodic. If it is, determine the amplitude.
b) The function illustrated in the graph is periodic because there is a repeating pattern of y-values on a regular basis. A periodic function usually has a maximum value and a minimum value every cycle. The amplitude is half the difference between the maximum and minimum values. In the graph shown, the maximum is 3 and the minimum is \(-1\). Therefore, the amplitude is \(\frac{3 - (-1)}{2}\), or 2 units.

Example 2

Predicting With Periodic Functions

Consider the periodic function shown.

a) What is the period of the function?
b) Determine \(f(2)\) and \(f(5)\).
c) Predict \(f(8), f(-10),\) and \(f(14)\).
d) What is the amplitude of the function?
e) Determine four \(x\)-values such that \(f(x) = 2\).

Solution

a) Select a convenient starting point, such as \((-7, 0)\). Move to the right until the pattern begins to repeat. This occurs at \((-1, 0)\). The period is equal to the horizontal length of this cycle, calculated by subtracting the \(x\)-coordinates. Thus, the period is \(-1 - (-7)\), or 6 units.

b) Read values from the graph: \(f(2) = 1\) and \(f(5) = 0\).

c) \(f(8) = f(2 + 6) = f(2)\) \(f(-10) = f(-10 + 6 + 6) = f(2)\) \(f(14) = f(14 - 6 - 6) = f(2)\) 
\(= 1\) \(= 1\) \(= 1\)

d) The maximum value is 3. The minimum value is \(-2\). The amplitude is \(\frac{3 - (-2)}{2}\), or 2.5 units.

e) From the graph, the value of \(f(0)\) is 2. Determine other \(x\)-values by adding the period to or subtracting the period from \(x = 0\). Two possible answers are \(x = 6, x = 12,\) and \(x = 18,\) or \(x = -6, x = -12,\) and \(x = -18\).
Example 3

Natural Gas Consumption in Ontario

The graph shows residential natural gas consumption in Ontario per month, beginning in January 2001. Data are obtained from Statistics Canada through its online E-STAT interactive tool.

a) Explain why the graph has this shape.

b) Do the data appear to be periodic? Justify your answer.

c) Assume that the consumption of natural gas in Ontario can be modelled using a periodic function. Determine the approximate maximum value, minimum value, and amplitude of this function.

d) Estimate the period of this function. Does this value make sense? Explain why.

e) Estimate the domain and range of the function.

f) Explain how the graph can be used to estimate the natural gas consumption in February 2011.

Solution

a) The consumption of natural gas in Ontario varies with the season. One expects consumption to be high in the winter months and low in the summer months.

b) The data are approximately periodic. The values do not exactly match from cycle to cycle.

c) A reasonable estimate for the maximum is 1 600 000 thousand cubic metres (1 600 000 000 m$^3$). The minimum is about 200 000 thousand cubic metres (200 000 000 m$^3$).

Amplitude $= \frac{1 600 000 - 200 000}{2}$

$= \frac{1 400 000}{2}$

$= 700 000$

The amplitude is about 700 000 thousand cubic metres (700 000 000 m$^3$).

d) From the graph, the period is about 12 months. This is reasonable. One expects the seasonal cycle for consumption of natural gas to be yearly.
Key Concepts

- A pattern that repeats itself regularly is periodic.
- A periodic pattern can be modelled using a periodic function.
- One repetition of a periodic pattern is called a cycle.
- The horizontal length of a cycle on a graph is called the period. The period may be in units of time or other units of measurement.
- A function is periodic if there is a positive number, $p$, such that $f(x + p) = f(x)$ for every $x$ in the domain of $f(x)$. The least value of $p$ that works is the period of the function.
- $f(x + np) = f(x)$, where $p$ is the period and $n$ is any integer.
- The amplitude of a periodic function is half the difference between the maximum value and the minimum value in a cycle.

Communicate Your Understanding

**C1** The population of a mining town has increased and decreased several times in the past few decades. Do you expect the population as a function of time to be periodic? Justify your answer.

**C2**

a) Consider a function such that $f(x + q) = -f(x)$. Sketch the graph of a simple function that follows this kind of relationship.

b) Is the function in part a) periodic? Justify your answer.

**C3** Consider the decimal expansion of the fraction $\frac{1}{7}$. If you graph each digit in the expansion on the vertical axis versus its decimal place on the horizontal axis, is the pattern periodic? Use a graph to support your answer.

e) Let $t$ represent the time, in months, and let $g$ represent the consumption of natural gas, in thousands of cubic metres. The domain is $\{t \in \mathbb{R}, 1 \leq t \leq 60\}$. Note that the lower bound of the domain is not 0. The data begin in January, which is the first month. The range is approximately $\{g \in \mathbb{R}, 200000 \leq g \leq 1600000\}$.

f) To obtain a reasonable estimate of gas consumption during the month of February, use the graph to find the consumption for each February shown. Take the average of these values. This is a reasonable estimate of the consumption predicted for February 2011.
A Practise

For help with questions 1 to 5, refer to Example 1.

1. Classify each graph as periodic or not periodic. Justify your answers.
   a) 
   
   b) 
   
   c) 
   
   d) 

2. Determine the amplitude and period for any graph in question 1 that is periodic.

3. Sketch four cycles of a periodic function with an amplitude of 5 and a period of 3.


5. Do your graphs in questions 3 and 4 match those of your classmates? Explain why or why not.

For help with questions 6 to 8, refer to Example 2.

6. A periodic function \( f(x) \) has a period of 8. The values of \( f(1) \), \( f(5) \), and \( f(7) \) are \(-3\), \(2\), and \(8\), respectively. Predict the value of each of the following. If a prediction is not possible, explain why not.
   a) \( f(9) \)
   b) \( f(29) \)
   c) \( f(63) \)
   d) \( f(40) \)

7. a) Sketch the graph of a periodic function, \( f(x) \), with a maximum value of 7, a minimum value of \(-1\), and a period of 5.
   b) Select a value \( a \) for \( x \), and determine \( f(a) \).
   c) Determine two other values, \( b \) and \( c \), such that \( f(a) = f(b) = f(c) \).

B Connect and Apply

8. Sunita draws a periodic function so that \( f(p) = f(q) \). Can you conclude that the period of the function is the difference between \( p \) and \( q \)? Justify your answer, including a diagram.

9. A navigation light on a point in a lake flashes 1 s on and 1 s off. After three flashes, the light stays off for an extra 2 s.
   a) Let 1 represent “on” and 0 represent “off.” Sketch a graph with time on the horizontal axis to represent the flashing of the light. Include three cycles.
   b) Explain why this pattern is periodic.
   c) What is the period of the pattern?
   d) What is the amplitude?
10. The people mover at an airport shuttles between the main terminal and a satellite terminal 300 m away. A one-way trip, moving at a constant speed, takes 1 min, and the car remains at each terminal for 30 s before leaving.
   a) Sketch a graph to represent the distance of the car from the main terminal with respect to time. Include four complete cycles.
   b) What is the period of the motion?
   c) What is the amplitude of the motion?

11. Which of the following values do you expect to follow a periodic pattern? Justify your answer for each case.
   a) the cost of 1 kg of tomatoes at the local supermarket at different times of the year
   b) the interest rate offered on an investment by a bank over a term of 5 years
   c) the percent of the moon’s face that is illuminated over several months
   d) the volume of air in your lungs during several minutes of normal breathing

12. Use Technology Sunspots are huge storms that take place on the sun. They can produce electromagnetic waves that interfere with radio, television, and other communication systems on Earth. Is the number of sunspots at any particular time random, or does the number follow a periodic pattern? Use the Internet to find a graph or table of sunspot activity over several decades. Inspect the data and decide whether the number of sunspots over time may be considered periodic. Justify your answer.

13. Is it possible for a periodic function to be either continuously increasing or continuously decreasing? Justify your answer, including a diagram.

14. While visiting the east coast of Canada, Ranouf notices that the water level at a town dock changes during the day as the tides come in and go out. Markings on one of the piles supporting the dock show a high tide of 3.3 m at 6:30 a.m., a low tide of 0.7 m at 12:40 p.m., and a high tide again at 6:50 p.m.
   a) Estimate the period of the fluctuation of the water level at the town dock.
   b) Estimate the amplitude of the pattern.
   c) Predict when the next low tide will occur.

15. The city of Quito, Ecuador, is located on the equator, about 6400 km from the centre of Earth. As Earth turns, Quito rotates about Earth’s axis. Consider midnight local time as the starting time and the position of Quito at that hour as the starting location. Let \( d \) represent the distance in a straight line from the starting location at time \( t \).
   a) Explain why the graph of \( d \) versus \( t \) will show a periodic pattern.
   b) What is the period of this motion?
   c) What is the amplitude of this motion?
   d) Suppose that you want to generate a table of values for \( d \) as a function of \( t \). Determine an appropriate trigonometric tool to use. Explain why it is the most appropriate tool for this problem. Describe how you would use the tool to generate the table of values.
16. The average monthly temperatures over 1 year in a given location usually follow a periodic pattern.

a) Estimate the maximum value, minimum value, and amplitude of this pattern for where you live.

b) What is the period of this pattern? Explain.

17. Describe a real-world pattern that you think might be periodic. Do not use a pattern that has already been used in this section. Trade patterns with a classmate. Perform an investigation to determine whether the pattern is periodic. If you determine that the pattern is periodic, determine the period and the amplitude. If it is not, explain why not.

18. **Chapter Problem** Randy connects his synthesizer to an oscilloscope and plays a B key that he knows produces a sound with frequency close to 500 cycles every second, or 500 Hz (hertz). The pattern is shown. Time, in seconds, is shown on the horizontal axis.

![Pattern Graph](image)

a) Explain how you know that the pattern is periodic.

b) What is the period?

c) Determine a relation between the period and the frequency of the note being played.

19. **Use Technology** Use dynamic geometry software to sketch a model of the carousel in the Investigate on page 284. Animate the point that represents Suzanne, and note how the measurements of angle and distance change during a revolution.

20. The hours and minutes of daylight on the first of each month of 2006 in Windsor, Ontario, are shown in the table.

<table>
<thead>
<tr>
<th>Date</th>
<th>Daylight (Hours:Minutes)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Jan. 1</td>
<td>9:09</td>
</tr>
<tr>
<td>Feb. 1</td>
<td>10:00</td>
</tr>
<tr>
<td>Mar. 1</td>
<td>11:14</td>
</tr>
<tr>
<td>Apr. 1</td>
<td>12:43</td>
</tr>
<tr>
<td>May 1</td>
<td>14:04</td>
</tr>
<tr>
<td>Jun. 1</td>
<td>15:04</td>
</tr>
<tr>
<td>Jul. 1</td>
<td>15:14</td>
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<td>Aug. 1</td>
<td>14:28</td>
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<tr>
<td>Sep. 1</td>
<td>13:10</td>
</tr>
<tr>
<td>Oct. 1</td>
<td>11:45</td>
</tr>
<tr>
<td>Nov. 1</td>
<td>10:21</td>
</tr>
<tr>
<td>Dec. 1</td>
<td>9:19</td>
</tr>
</tbody>
</table>

a) Explain why these data will show a periodic pattern over several years.

b) Predict the number of hours of daylight on May 1, 2010.

c) Predict the number of hours of daylight on September 15, 2008.

21. At the doctor’s office for a routine physical examination, Armand has his blood pressure checked. He notices that the pressure reaches a high value (systolic pressure) of 120 and a low value (diastolic pressure) of 80, measured in millimetres of mercury (mmHg). The doctor counts 18 pulse beats in 15 s. Is the blood pressure pattern periodic? Justify your answer.

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**Connections**

One of the oldest purely electronic instruments is the theremin, invented in 1919 by Leon Theremin, a Russian engineer. Players control the instrument’s sound by moving their hands toward or away from the instrument’s two antennas. One antenna controls the pitch of the sound; the other controls the volume. You have probably heard the eerie, gliding, warbling sounds of a theremin in science fiction or horror films.
Extend

22. A lighthouse beacon rotates through 360° every 12 s. The lighthouse is located 100 m off the shore of an island with a coastline of steep cliffs running north and south. As the light beam sweeps clockwise, starting from north (direction of 0°), it strikes some part of the cliff.

a) How long does it take the light beam to reach an angle of 30°? What is the distance travelled by the beam to the cliff at that time? Record time and distance in a table.

b) Repeat part a) for 60°, 90°, 120°, and 150°.

c) What happens to the distance as the beam approaches 180°?

d) After passing 180°, how long does it take until the beam strikes the cliff again at some point?

e) Use your table to sketch the graph of distance versus time for one revolution of the light.

f) Explain why your graph shows a periodic pattern. What is the period?

g) What is the amplitude of the pattern? Explain.

23. In some cases, the amplitude of a function decreases with time. An example is a function used to model the sound of a plucked guitar string. As time goes on, the sound becomes fainter and dies. This is known as a damped periodic function. An example is shown.

a) Construct a table of values of \( y \) versus \( x \), recording the maximum value for each cycle in the \( y \) column.

b) Draw a scatter plot for \( y \) versus \( x \).

c) What kind of model do the data appear to follow?

d) Use your knowledge from a previous chapter to construct the model.

e) Graph the model on your scatter plot. Comment on the fit.

24. Math Contest If the fraction \( \frac{5}{7} \) is written in expanded decimal form, what is the 100th digit after the decimal point?

A 1 B 4 C 2 D 8

25. Math Contest A number has the pattern 978675…0. How many digits does this number have?

A 14 B 16 C 18 D 20

26. Math Contest A sequence is created using the following rules.

- If the number is odd, the next number is found by adding 1 to the number and then dividing by 2.
- If the number is even, the next number is the number divided by 2.

If you start with the number 211, what is the 53rd number in the sequence?

A 27 B 3 C 1 D 100

27. Math Contest Of 50 students surveyed, 30 say they like algebra, 21 say they like trigonometry, and 8 say they like both. How many students do not like either algebra or trigonometry?

A 7 B 0 C 12 D 1